The Complete Proof Theory of Hybrid Systems

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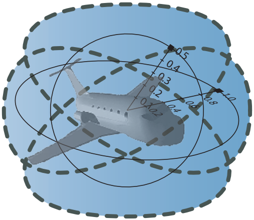
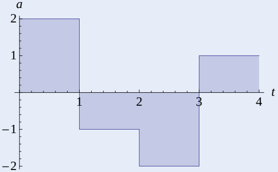


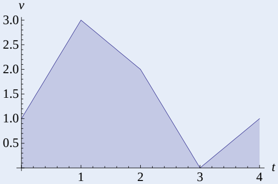
Hybrid Systems: e.g., Car Control

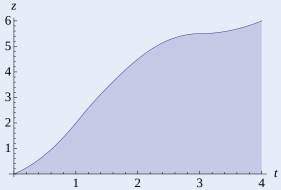
Challenge (Hybrid Systems)

Fixed rule describing state evolution

- Continuous dynamics (differential equations)
- Discrete dynamics (control decisions)







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1. More than computers: no `NullPointerException` $\not\Rightarrow$ safe

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2. More than physics: braking control $v^2 \leq 2b(M - x) \not\Rightarrow$ safe

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1. More than computers: no `NullPointerException` $\not\Rightarrow$ safe
2. More than physics: braking control $v^2 \leq 2b(M-x) \not\Rightarrow$ safe
3. Joint dynamics requires: $SB \geq \frac{v^2}{2b} + \frac{a^2\varepsilon^2}{2b} + \frac{a}{b}\varepsilon v + \frac{a}{2}\varepsilon^2 + \varepsilon v \ldots$

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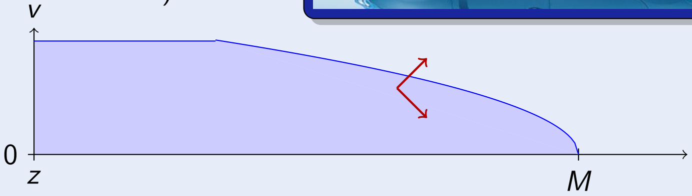

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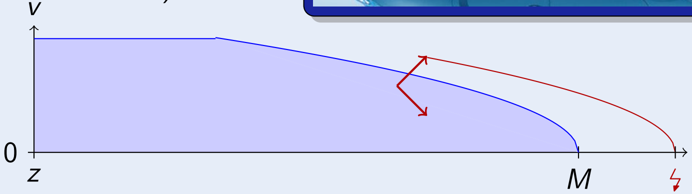

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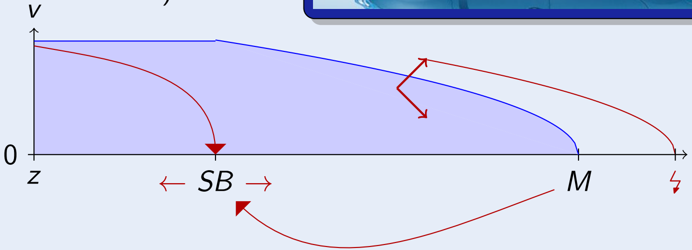

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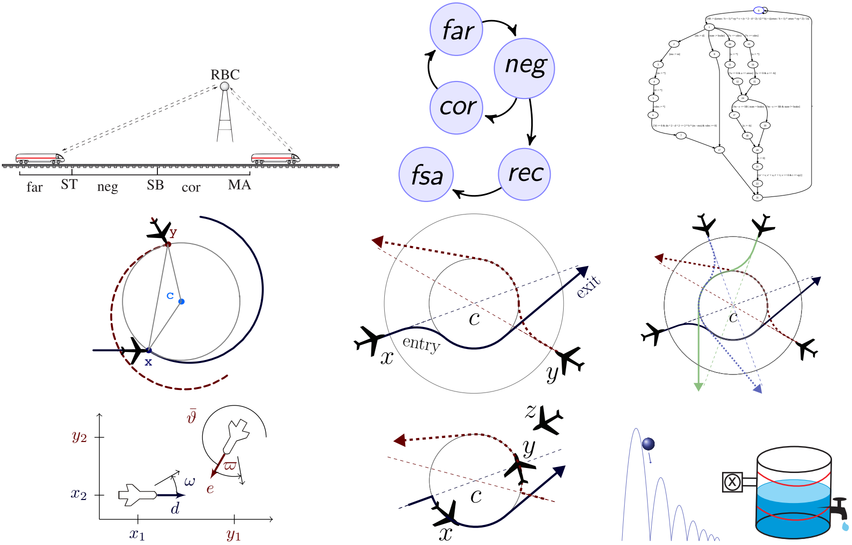
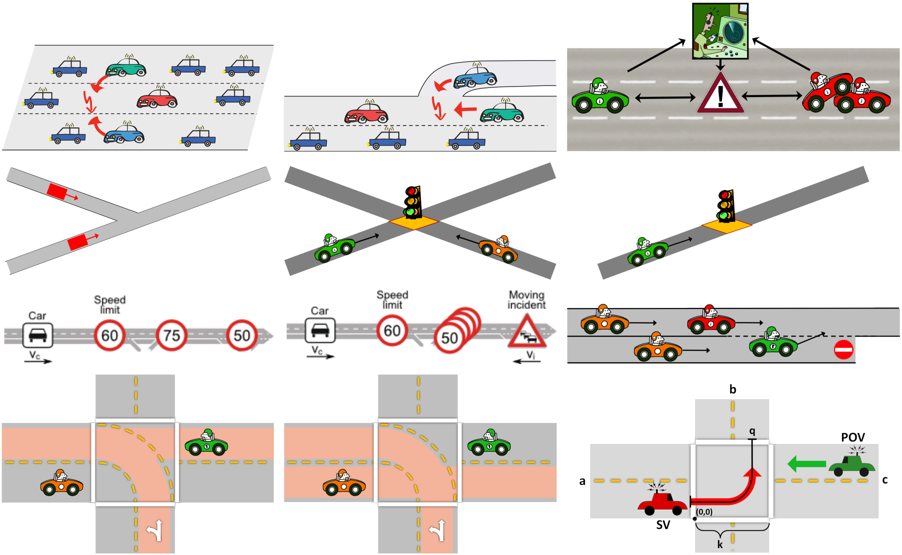
Successful Hybrid Systems Proofs



Successful Hybrid Systems Proofs



hybrid = continuous = discrete

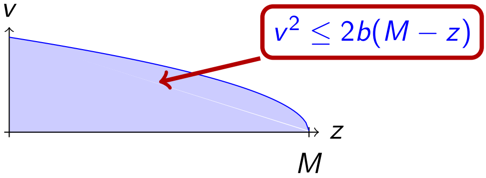
Differential Dynamic Logic for Hybrid Systems

differential dynamic logic

$\mathrm{d}\mathcal{L} = \qquad \mathrm{DL} + \mathrm{HP}$

differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$



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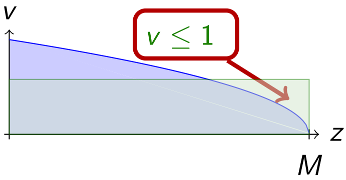

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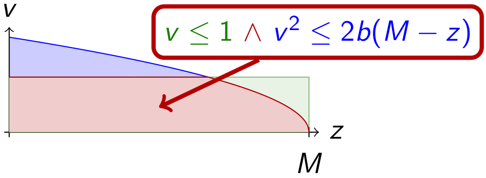

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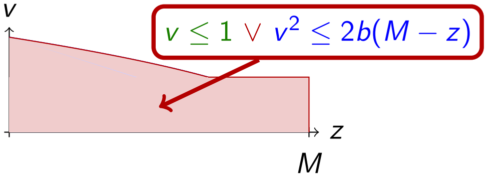

Differential Dynamic Logic for Hybrid Systems

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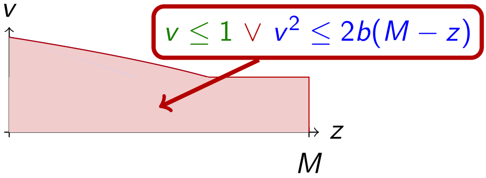
$\mathsf{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}}$

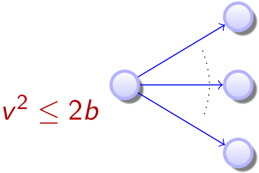
$\forall M A\, \exists S B \ \ldots$

$\forall t{\geq}0 \ \ldots$



differential dynamic logic

$d\mathcal{L} = \text{FOL}_{\mathbb{R}} +$



Differential Dynamic Logic for Hybrid Systems

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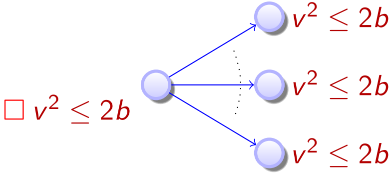
$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{ML}$



Differential Dynamic Logic for Hybrid Systems

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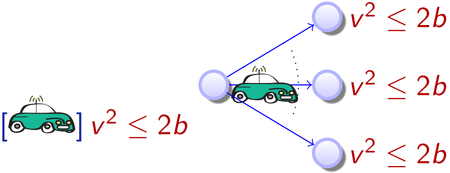
$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{DL}$

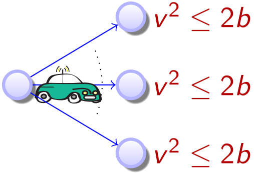


Differential Dynamic Logic for Hybrid Systems

differential dynamic logic

$d\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$

$[z'' = a]\, v^2 \leq 2b$



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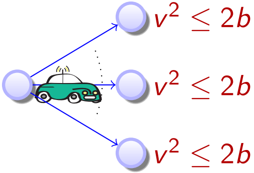
$$\mathrm{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$$

$$[\mathtt{if}(z > SB)\, a := -b;\ z'' = a]\, v^2 \leq 2b$$



Differential Dynamic Logic for Hybrid Systems

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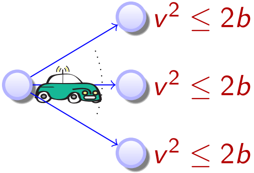
$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{HP}$

$$[\underbrace{\texttt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \le 2b$$



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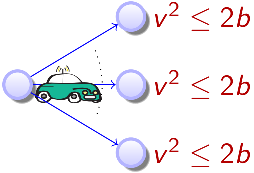
$\mathrm{d}\mathcal{L} = \mathrm{FOL}_{\mathbb{R}} + \mathrm{DL} + \mathrm{HP}$

$$\mathcal{C} \rightarrow [\underbrace{\mathtt{if}(z > SB)\, a := -b;\ z'' = a}_{\text{hybrid program}}]\, v^2 \leq 2b$$



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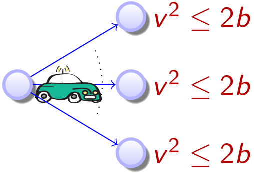

Initial condition

Differential Dynamic Logic for Hybrid Systems

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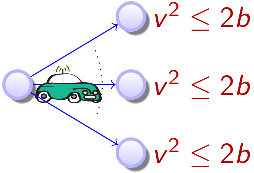

Initial condition

System dynamics

differential dynamic logic

$d\mathcal{L} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{HP}$

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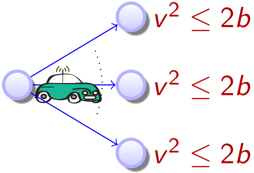


Initial condition

System dynamics

Post condition

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic d$\mathcal{L}$: Semantics

Definition (Hybrid program α)

$$
\begin{aligned}
\rho(x := \theta) &= \{(v,w) \;:\; w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v,v) \;:\; v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0),\varphi(r)) \;:\; \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha;\beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n\in\mathbb{N}} \rho(\alpha^n)
\end{aligned}
$$

Definition (d$\mathcal{L}$ Formula ϕ)

$$
\begin{aligned}
& v \models \theta_1 \geq \theta_2 && \text{iff} \quad [\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v \\
& v \models [\alpha]\phi && \text{iff} \quad w \models \phi \text{ for all } w \text{ with } (v,w) \in \rho(\alpha) \\
& v \models \langle\alpha\rangle\phi && \text{iff} \quad w \models \phi \text{ for some } w \text{ with } (v,w) \in \rho(\alpha) \\
& v \models \forall x\, \phi && \text{iff} \quad w \models \phi \text{ for all } w \text{ that agree with } v \text{ except for } x \\
& v \models \exists x\, \phi && \text{iff} \quad w \models \phi \text{ for some } w \text{ that agrees with } v \text{ except for } x
\end{aligned}
$$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

$$[:=] \quad [x := \theta]\phi(x) \leftrightarrow \phi(\theta)$$

$$[?] \quad [?H]\phi \leftrightarrow (H \rightarrow \phi)$$

$$['] \quad [x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi \qquad (y'(t) = f(y))$$

$$[\cup] \quad [\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \land [\beta]\phi$$

$$[;] \quad [\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$$

$$[^*] \quad [\alpha^*]\phi \leftrightarrow \phi \land [\alpha][\alpha^*]\phi$$

$$\mathrm{K} \quad [\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$$

$$\mathrm{I} \quad [\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$$

$$\mathrm{C} \quad [\alpha^*]\forall v{>}0\, (\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1)) \rightarrow \forall v\, (\varphi(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v))$$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

G $\dfrac{\phi}{[\alpha]\phi}$

MP $\dfrac{\phi \to \psi \quad \phi}{\psi}$

$\forall$ $\dfrac{\phi}{\forall x\, \phi}$

Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

G $\quad \dfrac{\phi}{[\alpha]\phi}$

MP $\quad \dfrac{\phi \to \psi \quad \phi}{\psi}$

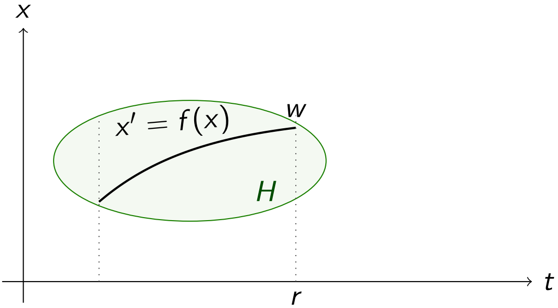
∀ $\quad \dfrac{\phi}{\forall x\, \phi}$

B $\quad \forall x\, [\alpha]\phi \to [\alpha]\forall x\, \phi \qquad (x \notin \alpha)$

V $\quad \phi \to [\alpha]\phi \qquad (FV(\phi) \cap BV(\alpha) = \emptyset)$

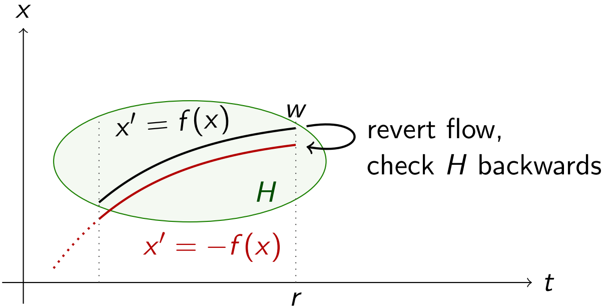
Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

[&] $[x' = f(x) \,\&\, H]\phi \leftrightarrow [x' = f(x)](\phi)$

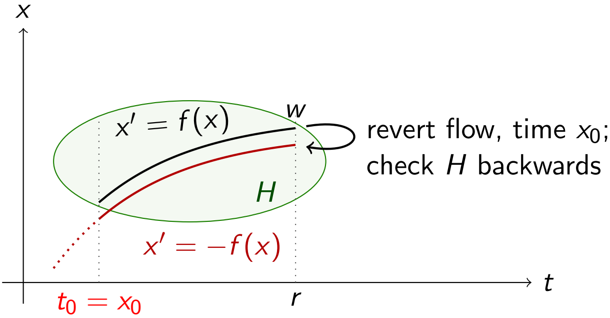


Differential Dynamic Logic d$\mathcal{L}$: Axiomatization

[&] $[x' = f(x) \,\&\, H]\phi$
$\leftrightarrow \quad [x' = f(x)]\big([x' = -f(x)](H) \to \phi\big)$

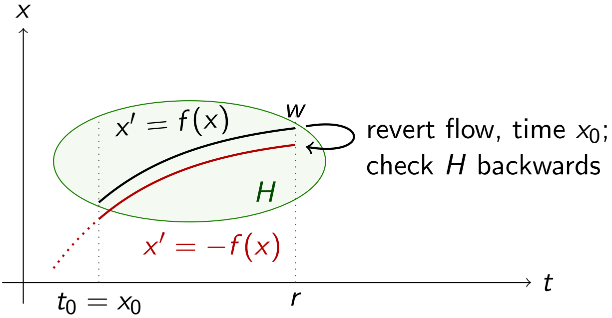


[&] $[x' = f(x)\,\&\,H]\phi$
$\leftrightarrow \forall t_0{=}x_0 [x' = f(x)]\big([x' = -f(x)](x_0 \geq t_0 \rightarrow H) \rightarrow \phi\big)$



“There and Back Again” Axiom of d$\mathcal{L}$

$$[\&] \quad [x' = f(x) \,\&\, H]\phi \leftrightarrow \forall t_0{=}x_0 [x' = f(x)]\big([x' = -f(x)](x_0 \geq t_0 \rightarrow H) \rightarrow \phi\big)$$



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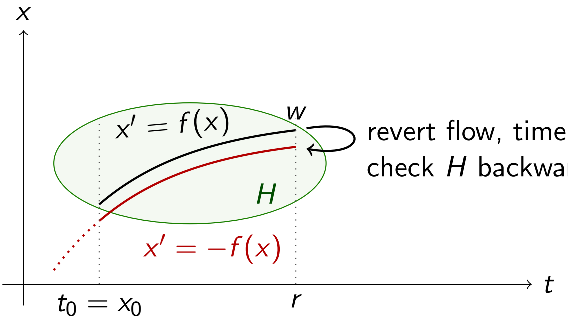

revert flow, time x_0;
check H backwards

Lemma

Evolution domain axiomatizable

Soundness

Theorem (Soundness)

dℒ *calculus is sound, i.e., all provable* dℒ *formulas are valid:*

$$\vdash \phi \text{ implies } \vDash \phi$$

What about the converse?

Continuous Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

Proof Outline 6p

$$\vDash \phi \textit{ implies } Taut_{FOD} \vdash \phi$$

Continuous Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

▸ Proof Outline 6p

$$\vDash \phi \text{ implies } Taut_{FOD} \vdash \phi$$

Corollary (Proof-theoretical Alignment)

proving hybrid systems = proving continuous dynamical systems!

Corollary (Compositionality)

hybrid systems can be verified by recursive decomposition

Discrete Completeness

Theorem (Relative Completeness / Continuous)

dℒ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

Proof Outline 6p

$$\vDash \phi \text{ implies } Taut_{FOD} \vdash \phi$$

Theorem (Relative Completeness / Discrete)

dℒ *calculus is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.*

Proof Outline +5p

$$\vDash \phi \text{ implies } Taut_{DL} \vdash \phi$$

Discrete Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.* ▸ Proof Outline 6p

$$\vDash \phi \text{ implies } Taut_{FOD} \vdash \phi$$

Theorem (Relative Completeness / Discrete)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.* ▸ Proof Outline +5p

$$\vDash \phi \text{ implies } Taut_{DL} \vdash \phi$$

Corollary (Complete Proof-theoretical Alignment)

hybrid = continuous = discrete

Discrete Completeness

Theorem (Relative Completeness / Continuous)

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Proof Outline +5p

$$\vDash \phi \text{ implies } Taut_{DL} \vdash \phi$$

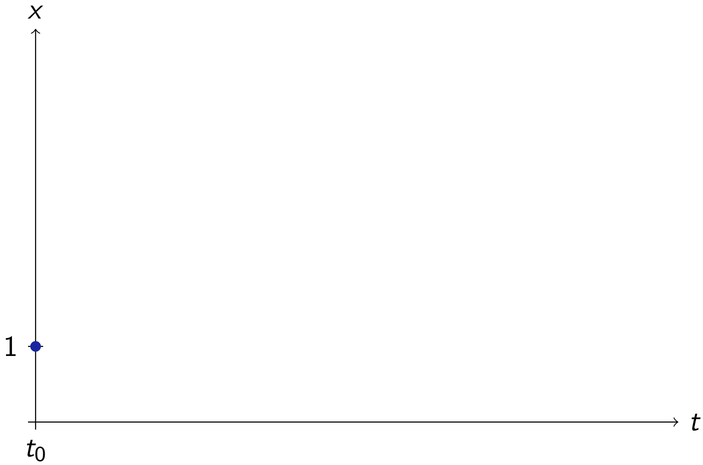
Corollary (Complete Proof-theoretical Alignment)

hybrid = continuous = discrete

Corollary (Interdisciplinary Integrability)

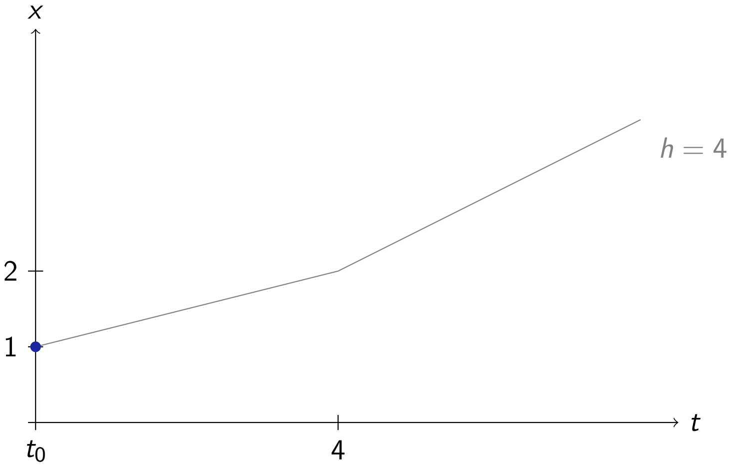
"Discrete computer science + continuous control are integrable"

$$[x' = \frac{x}{4}]F$$



Continuous Dynamics vs. Discrete Dynamics

$$[x' = \frac{x}{4}]F \qquad [(x := x + h\frac{x}{4})^*]F$$



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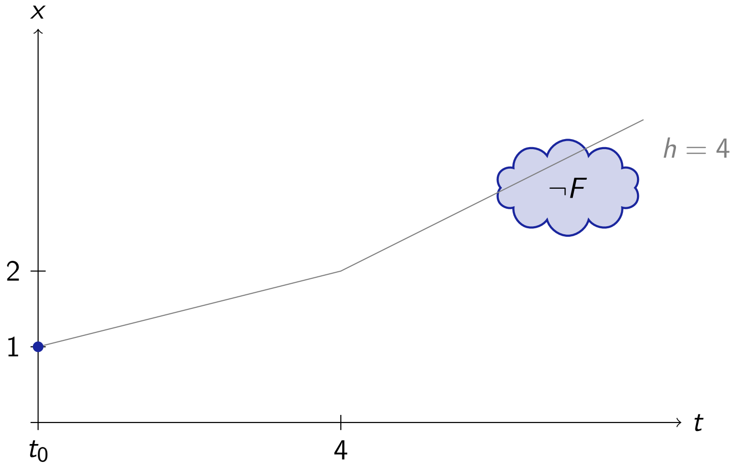

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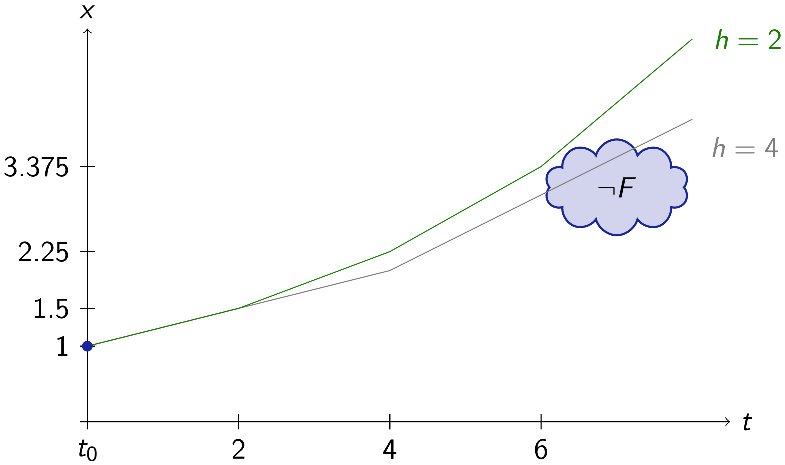

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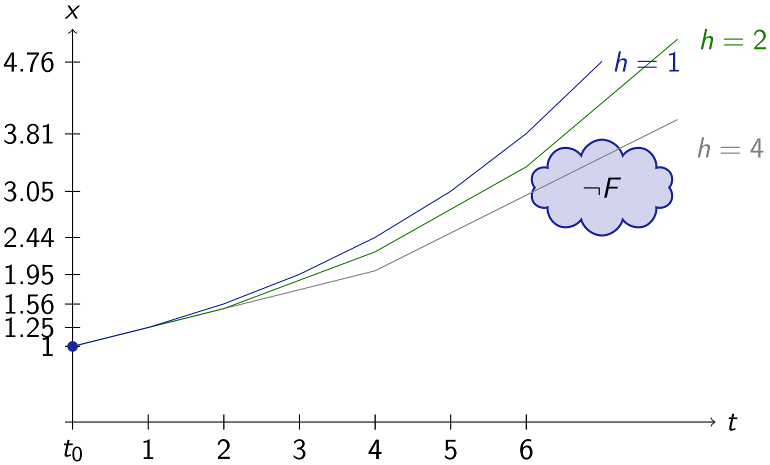

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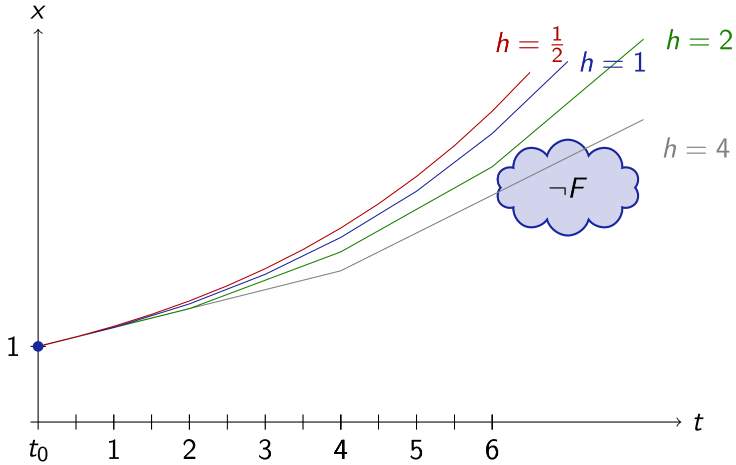
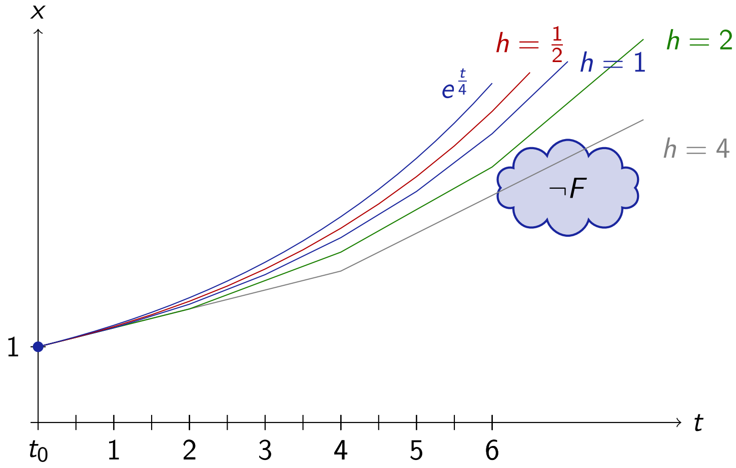


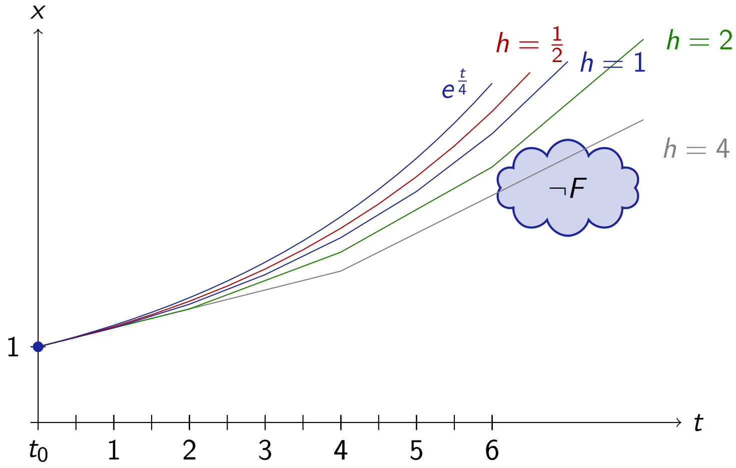
Continuous Dynamics vs. Discrete Dynamics

$$[x' = \frac{x}{4}]F \quad \text{vs.} \quad [(x := x + h\frac{x}{4})^*]F$$



Continuous Dynamics vs. Discrete Dynamics

$$[x' = \frac{x}{4}]F \quad \not\Rightarrow \quad [(x := x + h\frac{x}{4})^*]F$$



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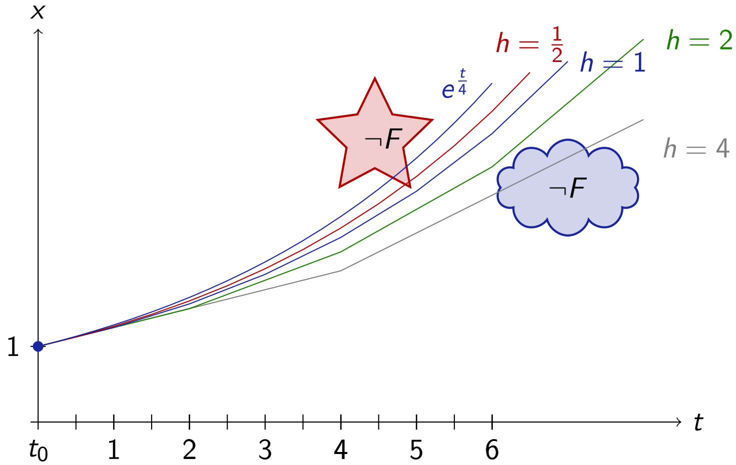

Discrete Euler Approximation Axiom $\overleftarrow{\Delta}$

$$\overleftarrow{\Delta} \quad [x' = f(x)]F \leftarrow \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*]F$$

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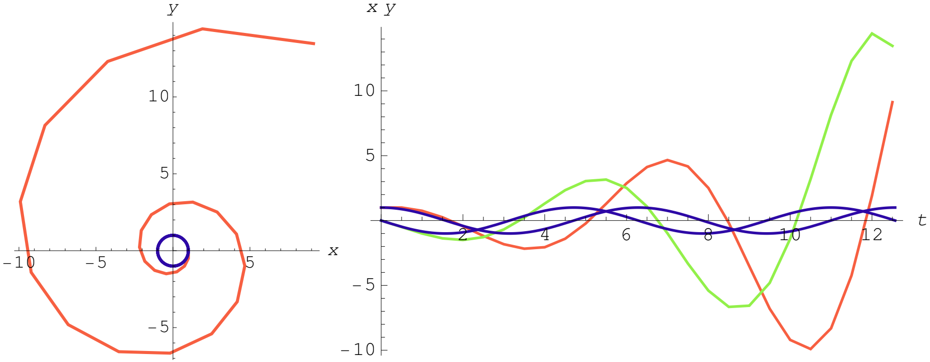
Example ()

$$\vDash x^2 + y^2 \leq 1.1 \rightarrow [x' = y, y' = -x]x^2 + y^2 \leq 1.1$$

$\overleftarrow{\Delta}$ $\quad [x' = f(x)]F$
$\leftarrow \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*]F$

Example (Insufficient, not global)

$$\vDash x^2 + y^2 \leq 1.1 \rightarrow [x' = y, y' = -x]x^2 + y^2 \leq 1.1$$

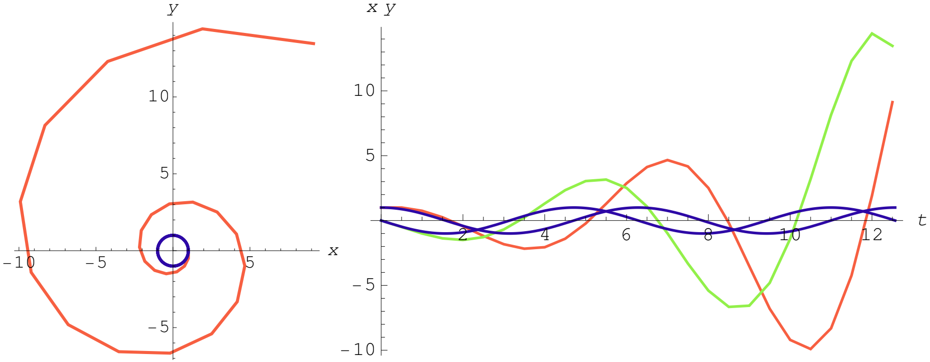


$$\overleftarrow{\Delta} \quad [x' = f(x)]F \leftarrow \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*]F$$

(closed)

Example (Unsound for open F, only in closure)

$$\nvDash x = 1 \land y = 0 \to [x' = y, y' = -x](x \leq 0 \to x^2 + y^2 > 1)$$



$$\overleftarrow{\Delta} \quad [x' = f(x)]F \leftarrow \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*]F \qquad \text{(closed)}$$

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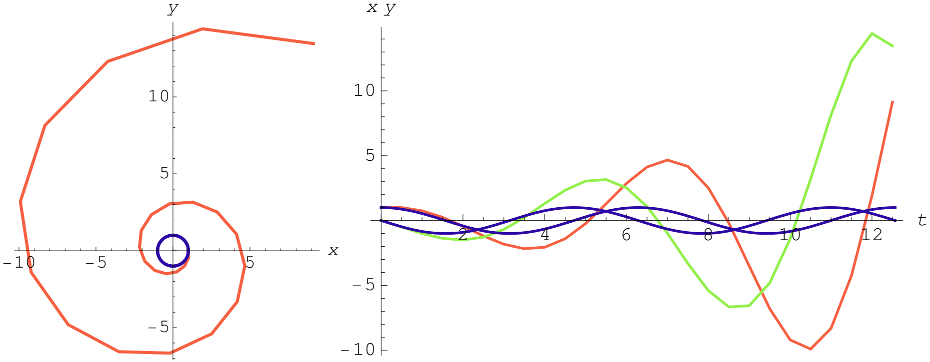


Discrete Euler Approximation Axiom $\overrightarrow{\Delta}$ $t' = -1$

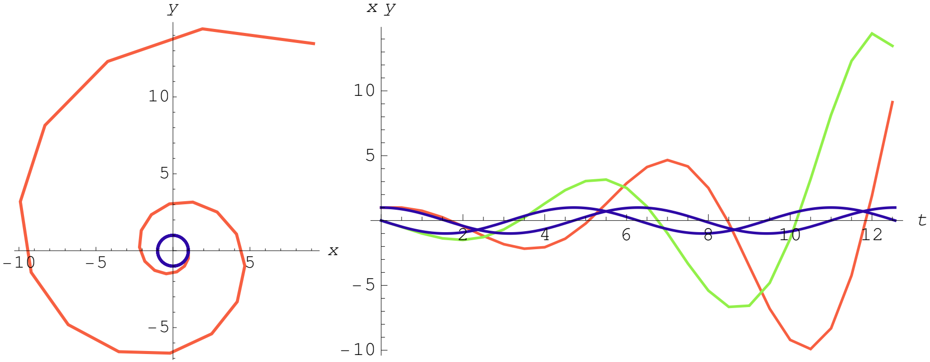
$$\overrightarrow{\Delta} \quad [x' = f(x)]F \\ \rightarrow \forall t{\geq}0\, \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*](t \geq 0 \rightarrow F)$$

Discrete Euler Approximation Axiom $\overrightarrow{\Delta}$

$$\overrightarrow{\Delta} \quad [x' = f(x)]F \rightarrow \forall t{\geq}0\, \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*](t \geq 0 \rightarrow F)$$

Example (Converse unsound for open F)

$$\not\models x = 1 \land y = 0 \rightarrow [x' = y, y' = -x](x \leq 0 \rightarrow x^2 + y^2 > 1)$$

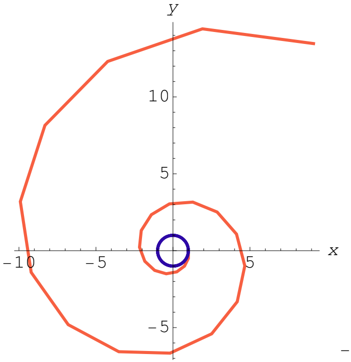


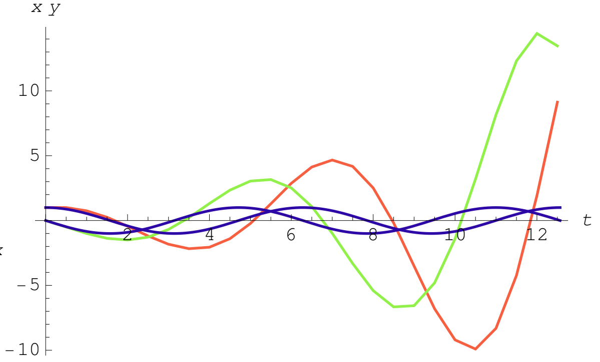
Discrete Euler Approximation Axiom $\overrightarrow{\Delta}$

$\overrightarrow{\Delta}$ $$[x' = f(x)]F \rightarrow \forall t{\geq}0\, \exists h_0{>}0\, \forall 0{<}h{<}h_0\, [(x := x + hf(x))^*](t \geq 0 \rightarrow F)$$ (open)

Example (Unsound for closed F, only holds in the limit)

$$\vDash x^2 + y^2 = 1 \rightarrow [x' = y, y' = -x]x^2 + y^2 = 1$$





Discrete Euler Approximation Axiom $\overleftrightarrow{\Delta}$ $t' = -1$

$$[x' = f(x)]F$$
$$\leftrightarrow \forall t{\geq}0 \exists \varepsilon_0{>}0 \forall 0{<}\varepsilon{<}\varepsilon_0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*]\big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$$

$[x' = f(x)]F$
$\leftrightarrow \forall t{\geq}0 \exists \varepsilon_0{>}0 \forall 0{<}\varepsilon{<}\varepsilon_0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*]\big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$

Example ()

$$\vDash x^2 + y^2 < 1.1 \to [x' = y, y' = -x]x^2 + y^2 < 1.1$$

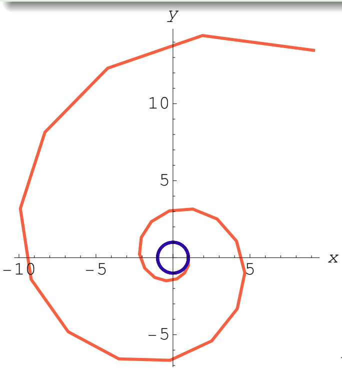
$\mathcal{R}$ Discrete Euler Approximation Axiom $\overleftrightarrow{\Delta}$ $\quad t' = -1$

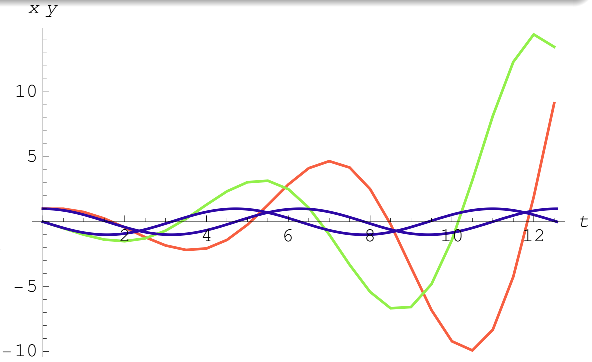
$[x' = f(x)]F$

$\leftrightarrow \forall t{\geq}0\,\exists\varepsilon_0{>}0\,\forall 0{<}\varepsilon{<}\varepsilon_0\,\exists h_0{>}0\,\forall 0{<}h{<}h_0\,[(x := x{+}hf(x))^*]\big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$

Example (Insufficient for closed F)

$$\vDash x^2 + y^2 \leq 1 \to [x' = y, y' = -x]x^2 + y^2 \leq 1$$





$[x' = f(x)]F$ *(open)*

$\leftrightarrow \forall t{\geq}0 \exists \varepsilon_0{>}0 \forall 0{<}\varepsilon{<}\varepsilon_0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*] \big(t{\geq}0 \rightarrow \neg \mathcal{U}_\varepsilon(\neg F)\big)$

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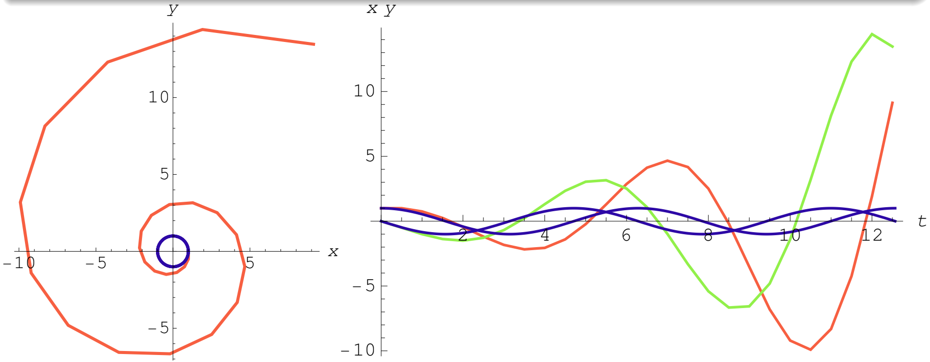
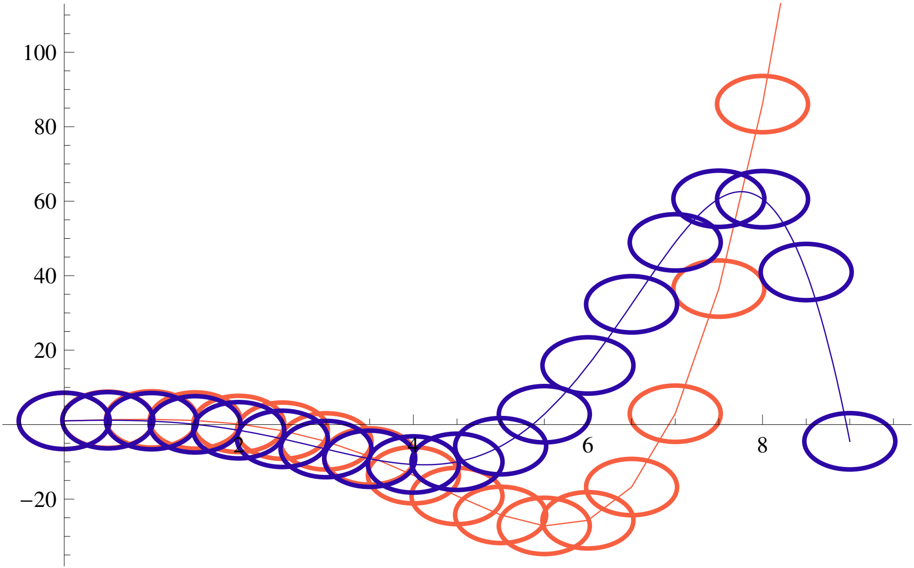
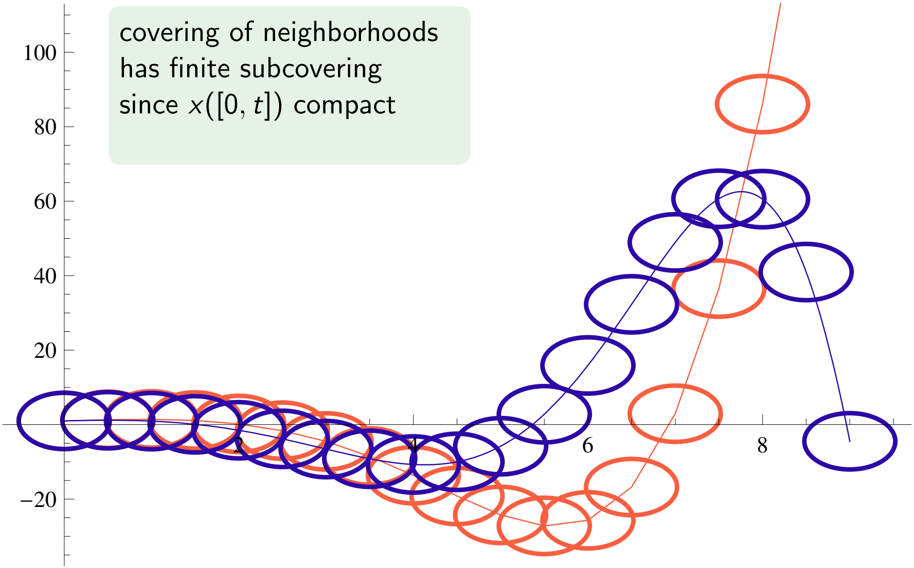


Partial Covering for Solution and Approximation



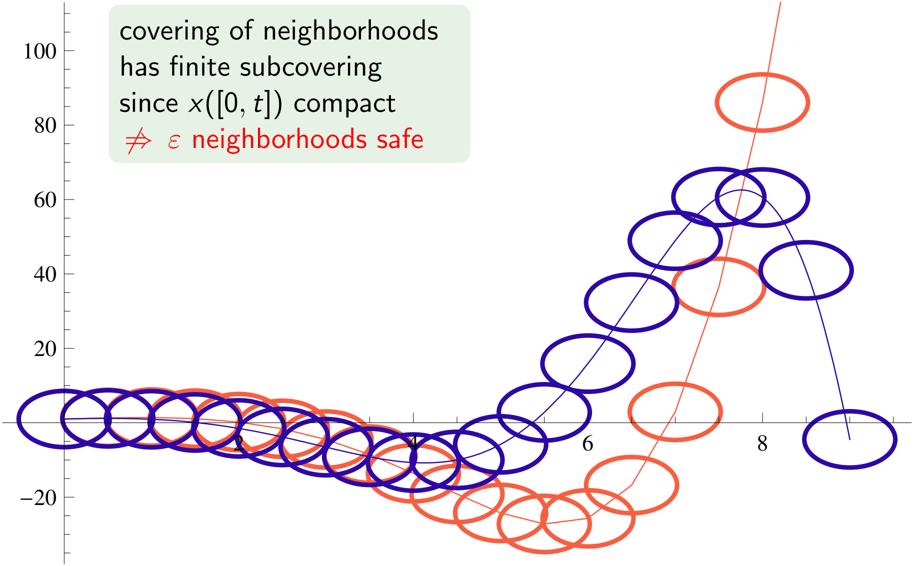
Partial Covering for Solution and Approximation



covering of neighborhoods has finite subcovering since $x([0,t])$ compact

Partial Covering for Solution and Approximation

covering of neighborhoods
has finite subcovering
since $x([0,t])$ compact
$\not\Rightarrow$ ε neighborhoods safe



$$[x' = f(x)]F \qquad (open)$$

$$\leftrightarrow \forall t{\geq}0 \exists \varepsilon_0{>}0 \forall 0{<}\varepsilon{<}\varepsilon_0 \exists h_0{>}0 \forall 0{<}h{<}h_0 [(x := x{+}hf(x))^*] \big(t{\geq}0 \to \neg\mathcal{U}_\varepsilon(\neg F)\big)$$

Example (Insufficient for closed F)

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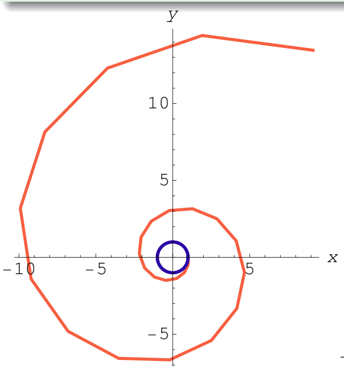

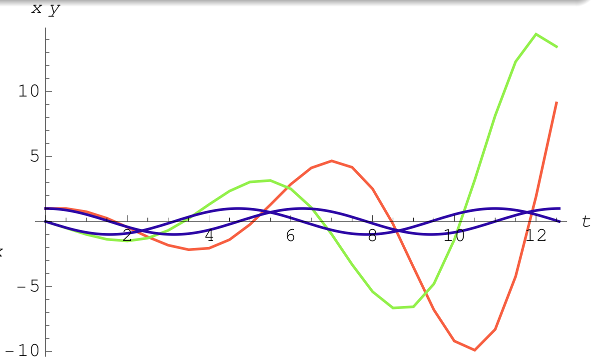

Closed Discrete Completeness (derivable)

$\mathring{U}$ $[x' = f(x)]F \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)]\mathcal{U}_{\check{\varepsilon}}(F)$ $(\Leftarrow \text{B,V,G,K})$

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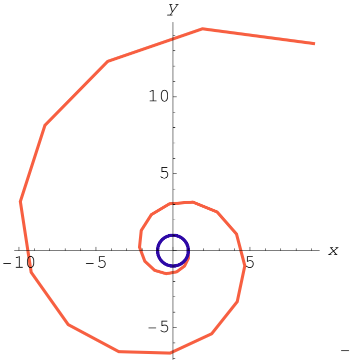
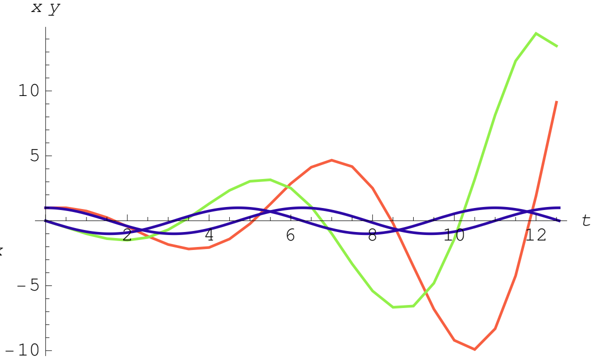
Example (Closed ⇝ Quantified Open)

$$\vDash x^2 + y^2 \leq 1 \rightarrow [x' = y, y' = -x]x^2 + y^2 \leq 1$$





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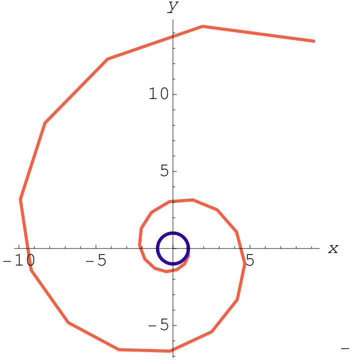
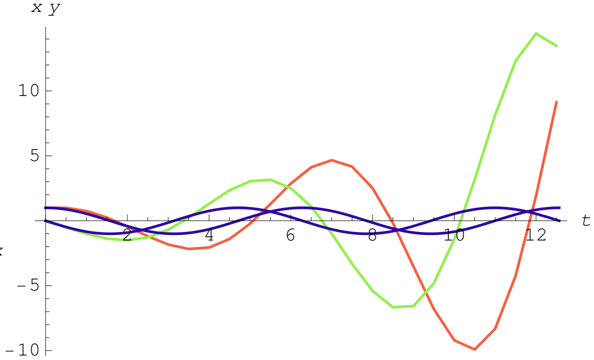
Example (Closed $\leadsto$ Quantified Open)

$$\vDash x^2 + y^2 \le 1 \rightarrow [x' = y, y' = -x]\forall \check{\varepsilon}{>}0\, x^2 + y^2 < 1 + \check{\varepsilon}$$





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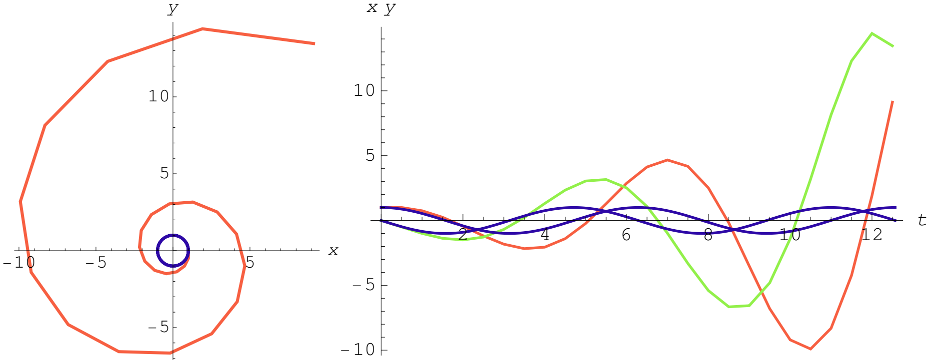


Locally Closed Discrete Completeness: $O \land C$ (derivable)

Example (Locally Closed $\rightsquigarrow$ Open, Closed)

$$\vDash O \land C \to [x' = y, y' = -x](O \land C)$$

Locally Closed Discrete Completeness: $O \land C$ (derivable)

$[]\land \quad [\alpha](O \land C) \leftrightarrow [\alpha]O \land [\alpha]C \qquad (\Leftarrow \text{K})$

Example (Locally Closed $\leadsto$ Open, Closed)

$$\vDash O \land C \rightarrow [x' = y, y' = -x](O \land C)$$

Locally Closed Discrete Completeness: $O \wedge C$ (derivable)

$[]\wedge \quad [\alpha](O \wedge C) \leftrightarrow [\alpha]O \wedge [\alpha]C \qquad (\Leftarrow \text{K})$

Example (Locally Closed $\rightsquigarrow$ Open, Closed)

$$\vDash O \wedge C \rightarrow [x' = y, y' = -x]O \wedge [x' = y, y' = -x]C$$

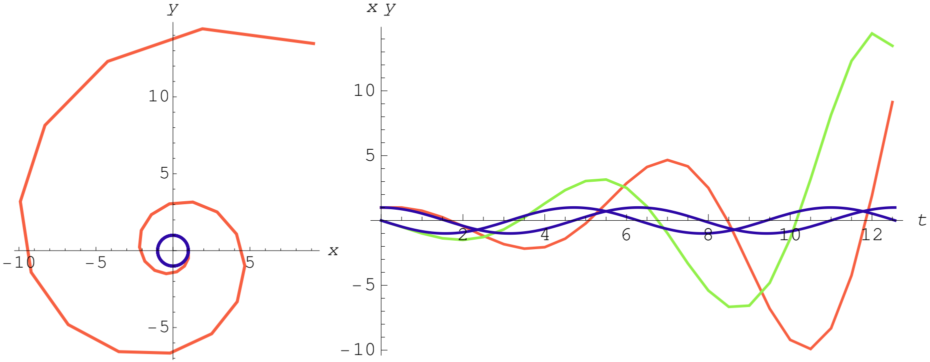
0-Order Discrete Completeness (derivable)

$$\check{U} \quad [x' = f(x)](O \lor C) \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)](O \lor \mathcal{U}_{\check{\varepsilon}}(C)) \qquad (\Leftarrow \text{ B,V,G,K})$$

$\check{U}$ $[x' = f(x)](O \lor C) \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)](O \lor \mathcal{U}_{\check{\varepsilon}}(C))$ $(\Leftarrow$ B,V,G,K)

Example ((Open ∨ Closed) ⇝ Quantified Open)

$$\vDash O \lor C \to [x' = y, y' = -x](O \lor C)$$



$\check{U}$ $[x' = f(x)](O \vee C) \leftrightarrow \forall \check{\varepsilon}{>}0\, [x' = f(x)](O \vee \mathcal{U}_{\check{\varepsilon}}(C))$ $(\Leftarrow$ B,V,G,K$)$

Example ((Open ∨ Closed) ⇝ Quantified Open)

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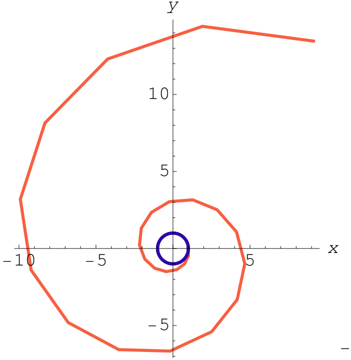

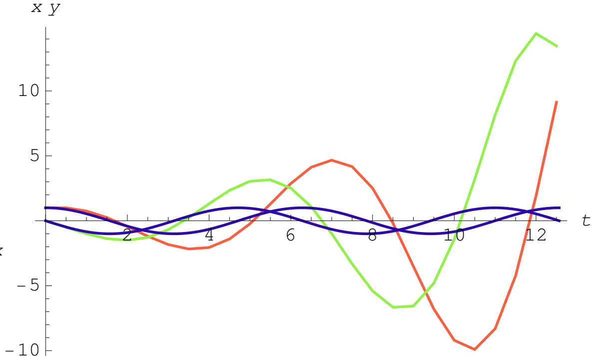

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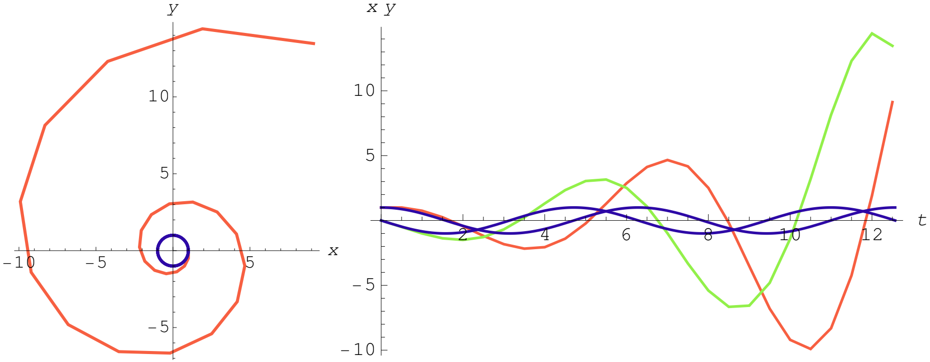
$$\vDash O \vee C \to \forall \check{\varepsilon}{>}0\, [x' = y, y' = -x](O \vee \mathcal{U}_{\check{\varepsilon}}(C))$$



Discrete Completeness

Theorem (Relative Completeness / Continuous)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to differential equations.*

Proof Outline 6p

$$\vDash \phi \text{ implies } Taut_{FOD} \vdash \phi$$

Theorem (Relative Completeness / Discrete)

d$\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to discrete dynamics.*

Proof Outline +5p

$$\vDash \phi \text{ implies } Taut_{DL} \vdash \phi$$

Proof Sketch.

Talked about 0-Order

Paper proves $\forall, \exists$ ...

Paper proves $[\alpha], \langle\alpha\rangle$ with hybrid system α ...

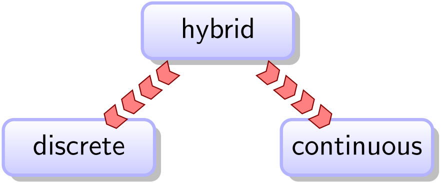
Paper proves nesting ... □

Equi-expressibility

Theorem (Equi-expressibility)

d$\mathcal{L}$ (constructively) expressible in FOD and in DL:

$$\forall\phi\ \exists\phi^\flat \in FOD \quad \vDash \phi \leftrightarrow \phi^\flat$$
$$\forall\phi\ \exists\phi^\# \in DL \quad \vDash \phi \leftrightarrow \phi^\#$$



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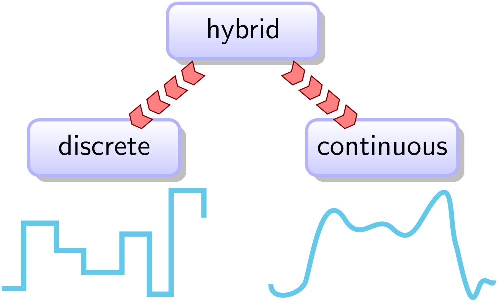

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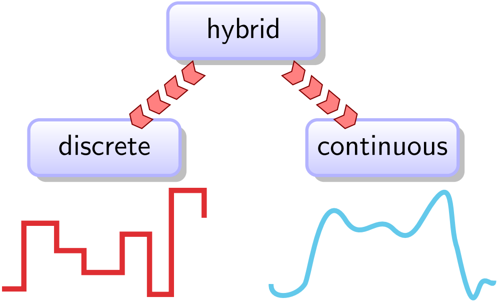

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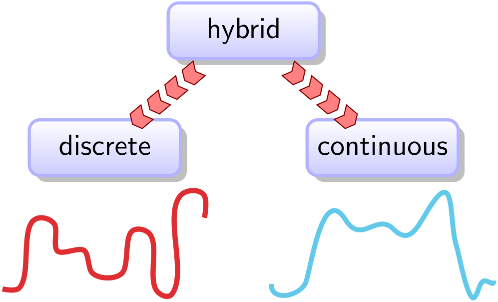

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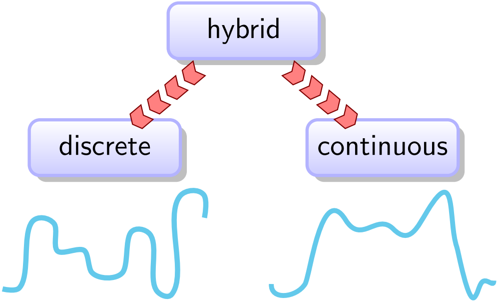

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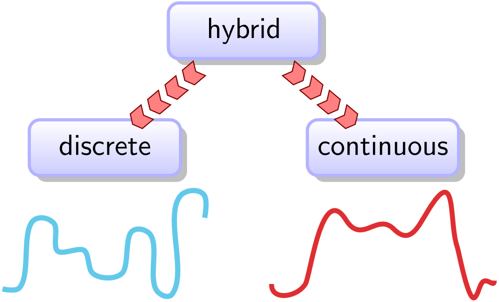

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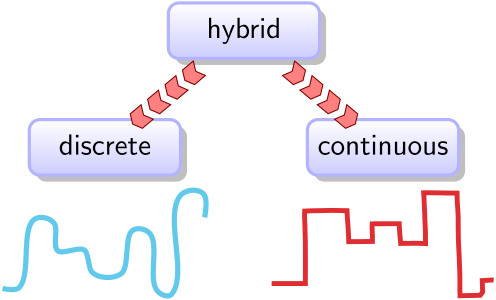

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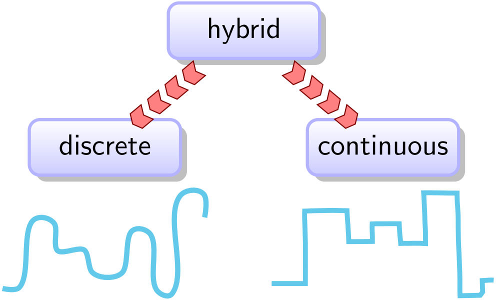


Relative Decidability

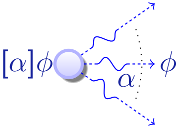
Theorem (Relative Decidability)

Validity of $d\mathcal{L}$ *sentences is decidable relative to FOD or DL.*

The Complete Proof Theory of Hybrid Systems

differential dynamic logic

$d\mathcal{L} = \mathrm{DL} + \mathrm{HP}$



proof-theoretical alignment

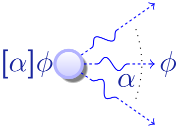
hybrid $=$ continuous $=$ discrete

- Differential dynamic logic
- Hybrid systems
- Axiomatization
- Complete / continuous
- Complete / discrete
- Constructive
- Euler $\overleftrightarrow{\Delta}$
- Perfect proof lifting
- Computer $+$ Control Science

The Complete Proof Theory of Hybrid Systems

differential dynamic logic

$d\mathcal{L} = DL + HP$



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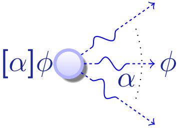
- Differential dynamic logic
- Hybrid systems
- Axiomatization
- Complete / continuous
- Complete / discrete
- Constructive
- Euler $\overleftrightarrow{\Delta}$
- Perfect proof lifting
- Computer + Control Science
- Discrete invariant

The Complete Proof Theory of Hybrid Systems

differential dynamic logic

$d\mathcal{L} = DL + HP$



proof-theoretical alignment

hybrid = continuous = discrete

- Differential dynamic logic
- Hybrid systems
- Axiomatization
- Complete / continuous
- Complete / discrete
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- Perfect proof lifting
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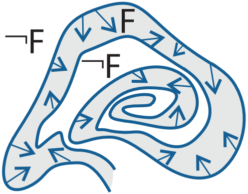
$\rightsquigarrow$ differential invariant



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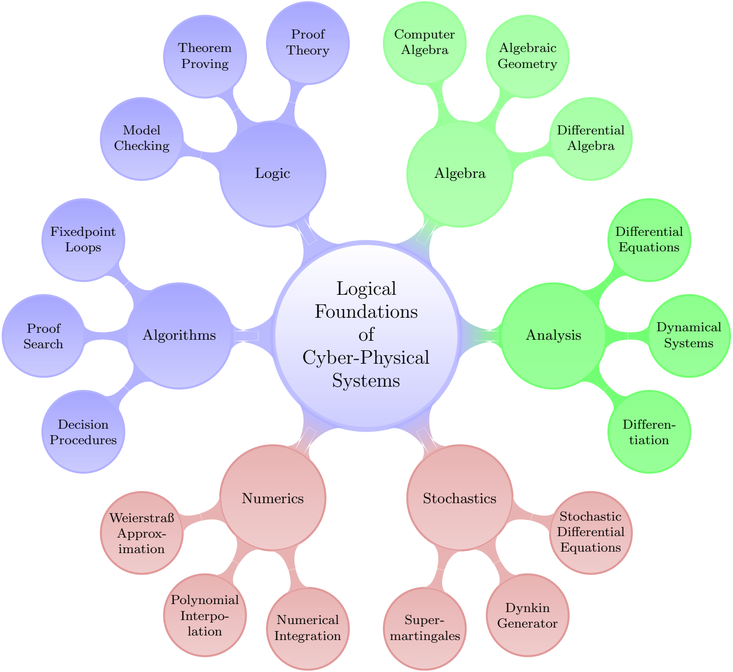
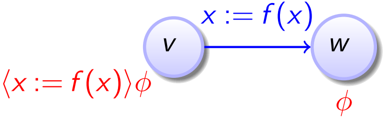
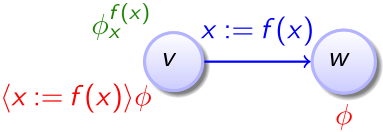

Logical Foundations of Cyber-Physical Systems
Logic
Model Checking
Theorem Proving
Proof Theory
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Analysis
Differential Equations
Dynamical Systems
Differen-tiation
Stochastics
Stochastic Differential Equations
Dynkin Generator
Super-martingales
Numerics
Numerical Integration
Polynomial Interpo-lation
Weierstraß Approx-imation
Algorithms
Fixedpoint Loops
Proof Search
Decision Procedures

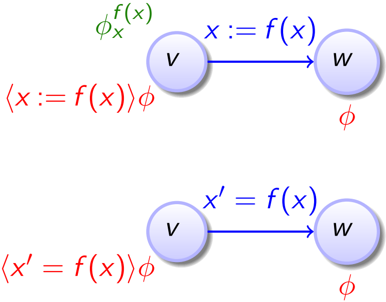
Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



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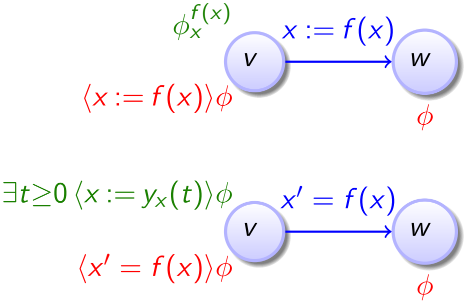
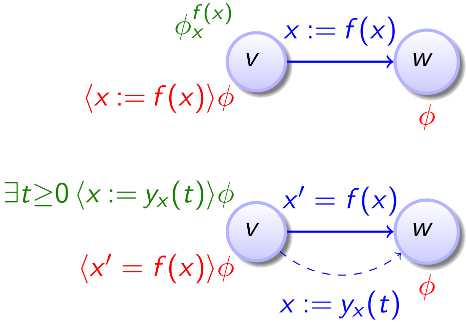
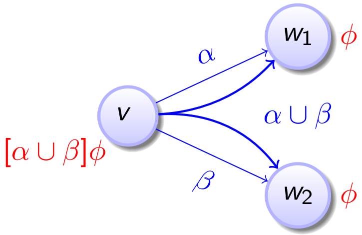


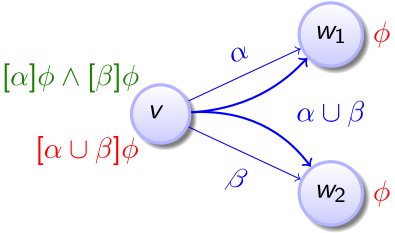
Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



Proof by Symbolic Decomposition



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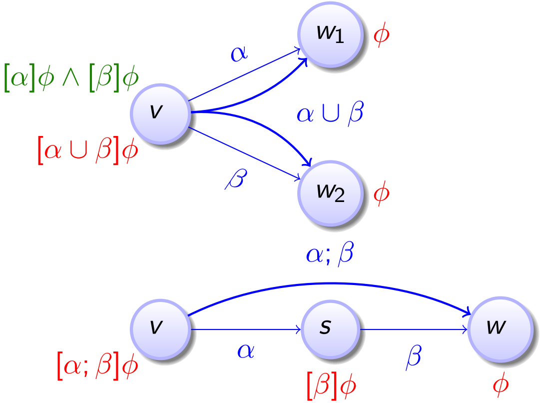
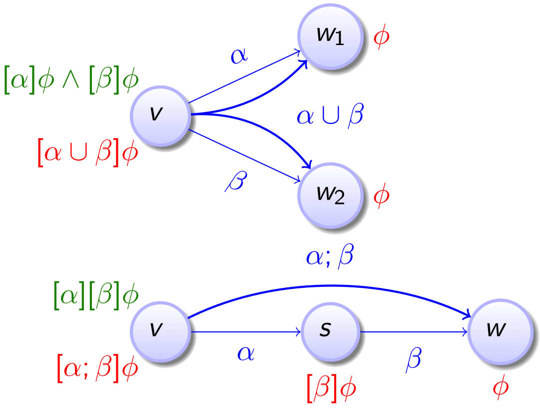


Proof by Symbolic Decomposition



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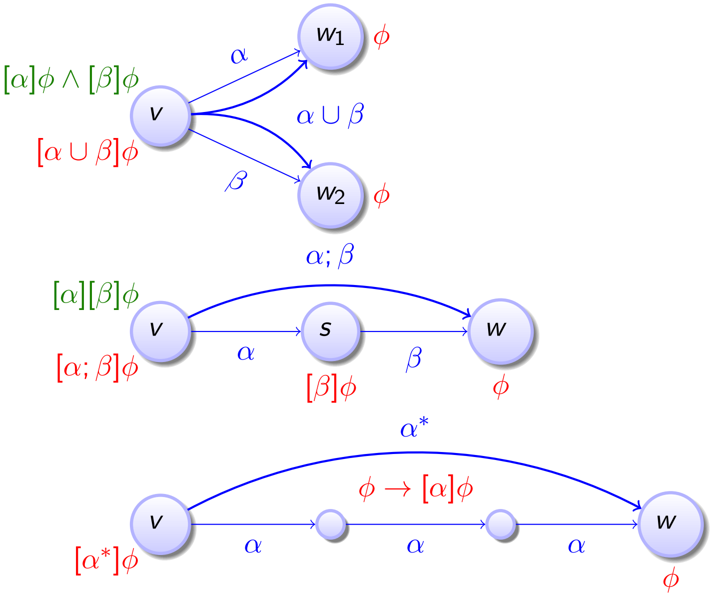

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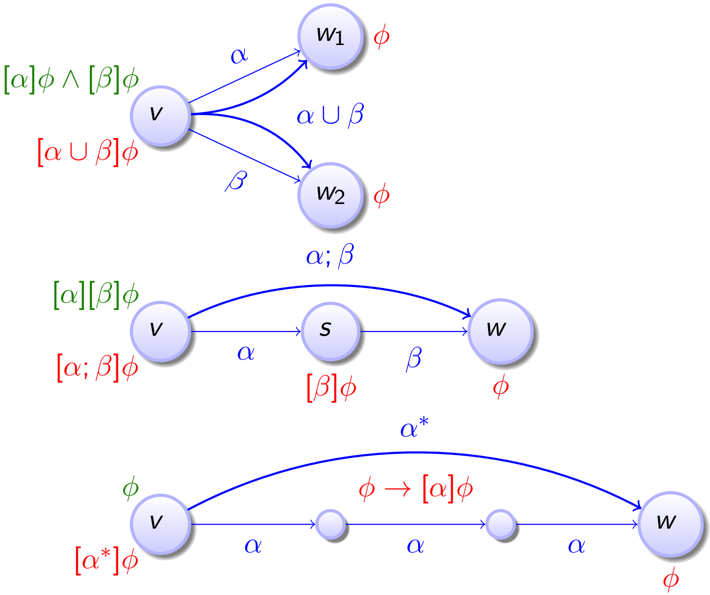

Proof by Symbolic Decomposition