Advances in Reachability Analysis with Applications to Safety Verification of Vehicle Control Systems

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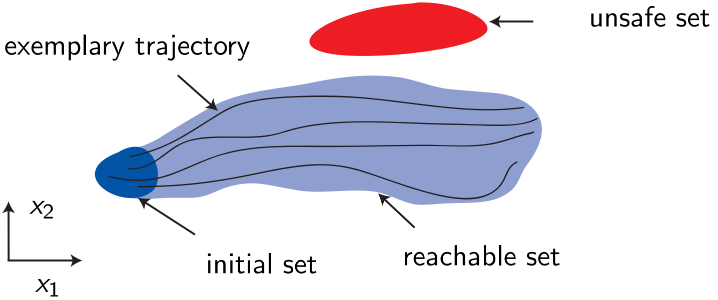
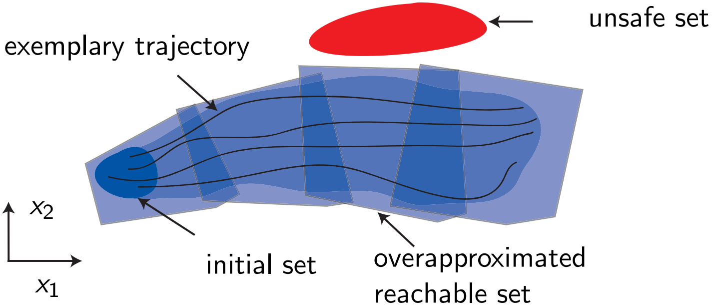
Safety Verification Using Reachable Sets



- System is safe, if no trajectory enters the unsafe set.

Safety Verification Using Reachable Sets



- System is safe, if no trajectory enters the unsafe set.
- Overapproximated system is safe $\rightarrow$ real system is safe.

Main Innovations

Consideration of Time-Varying Parameters for Linear Systems

There is much work for linear time invariant (LTI) Systems; a wrapping-free algorithm exists [A. Girard, C. Le Guernic, O. Maler; HSCC 2006].

Here: The system matrix is uncertain and time-varying.

Novel Linearization Approach for Nonlinear Systems

Before: The linearization error is considered by an additional uncertain input.

Here: The linearization error is considered by adding parameter uncertainties.

Continuization of Hybrid Systems

Before: Hybrid dynamics requires intersection of reachable sets with guard sets.

Here: The intersection can be eliminated by temporarily enlarging the set of uncertain parameters.

Considered Class of Systems

Linear systems with uncertain time varying parameters

$$\dot{x}(t) = A(t)x(t) + u(t),$$

where $A : \mathbb{R}^+ \rightarrow \mathcal{A}$, $u : \mathbb{R}^+ \rightarrow \mathcal{U}$ are piecewise continuous, and $\mathcal{A} \subset \mathbb{R}^{n \times n}$, $\mathcal{U} \subset \mathbb{R}^n$. For reachability analysis, we consider all possible functions $A(t)$ and $u(t)$.

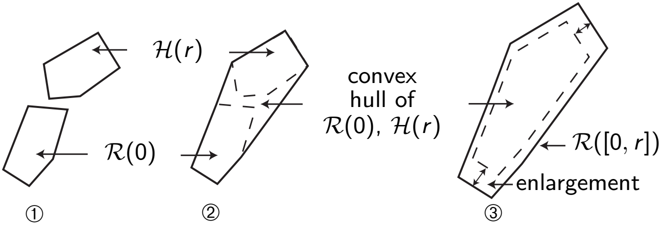
Example:

$$\mathcal{A} = \begin{pmatrix} [-1.05, -0.95] & [-4.05, -3.95] \\ [3.95, 4.05] & [-1.05, -0.95] \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} [-0.05, 0.05]$$

Overview of Reachable Set Computation

1. Compute reachable set $\mathcal{H}(r)$ at time r when there is no input.
   Input not yet considered.
2. Obtain convex hull of initial set $\mathcal{R}(0)$ and $\mathcal{H}(r)$.
   Curvature of trajectories not yet considered.
3. Enlarge reachable set to account for (1) uncertain inputs, (2) curvature of trajectories.
4. Continue with further time intervals $[kr, (k+1)r]$, $k \in \mathbb{N}$.



Peano Baker Series

Superposition principle: First, consider only the initial state solution

$$x(t) = \Phi(A(\tau), t)x_0,$$

where $\Phi(A(\tau), t)$ is referred to as the Peano Baker Series.

Peano Baker Series

$$\begin{aligned}\Phi(A(\tau), t) = & \mathrm{I} + \int_0^t A(\sigma_1)d\sigma_1 + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2)\, d\sigma_2\, d\sigma_1 \\ & + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2) \int_0^{\sigma_2} A(\sigma_3)\, d\sigma_3\, d\sigma_2\, d\sigma_1 + \dots\end{aligned}$$

How to compute the set $\{\Phi(A(\tau), t) | A(\tau) \in \mathcal{A}\}$*?*

Overapproximation of the Peano Baker Series

1. Time discretization: $\int_0^t A(\sigma_i)d\sigma_i \approx \sum_{l_i=1}^{k} A(l_i\Delta)\Delta,\ t = k\Delta$ (Riemann integration).

Approximate $\Phi(A(\tau), t)$ iteratively as

$$\tilde{\Phi}_1(A(\tau), k, \Delta) = \mathtt{I} + \sum_{l_1=1}^{k} A(l_1\Delta)\Delta,$$

$$\tilde{\Phi}_i(A(\tau), k, \Delta) = \tilde{\Phi}_{i-1}(t, \Delta) + \sum_{l_i=1}^{k} \ldots \sum_{l_1=1}^{l_2} \left(\prod_{q=1}^{i} A(l_q\Delta) \right) \Delta^i,$$

Reminder: $$\Phi(A(\tau), t) = \underbrace{\overbrace{\mathtt{I} + \int_0^t A(\sigma_1)d\sigma_1}^{i=1} + \int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2)\, d\sigma_2\, d\sigma_1}_{i=2} + \ldots$$

Overapproximation of the Peano Baker Series

1. Time discretization: $\int_0^t A(\sigma_i)d\sigma_i \approx \sum_{l_i=1}^k A(l_i\Delta)\Delta$, $t = k\Delta$ (Riemann integration).
2. Replace concrete matrices by sets of matrices.

Approximate $\Phi(A(\tau), t)$ iteratively as

$$\tilde{\Phi}_1(A(\tau), k, \Delta) = \mathrm{I} + \underbrace{\sum_{l_1=1}^{k} A(l_1\Delta)\Delta}_{\in \bigoplus_{l_1=1}^{k} \mathcal{A}\Delta},$$

$$\tilde{\Phi}_i(A(\tau), k, \Delta) = \tilde{\Phi}_{i-1}(t, \Delta) + \underbrace{\sum_{l_i=1}^{k} \dots \sum_{l_1=1}^{l_2} \left(\prod_{q=1}^{i} A(l_q\Delta) \right) \Delta^i}_{\in \bigoplus_{l_i=1}^{k} \dots \bigoplus_{l_1=1}^{l_2} \mathcal{A}^i \Delta^i},$$

where $\oplus$ represents the Minkowski addition: $\mathcal{A} \oplus \mathcal{B} = \{A + B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

Overapproximation of the Peano Baker Series

1. Time discretization: $\int_0^t A(\sigma_i)d\sigma_i \approx \sum_{l_i=1}^k A(l_i\Delta)\Delta$, $t = k\Delta$ (Riemann integration).
2. Replace concrete matrices by sets of matrices.
3. Apply distributivity of convex matrix sets: $a\mathcal{A} \oplus b\mathcal{A} = (a+b)\mathcal{A}$

Approximate $\Phi(A(\tau), t)$ iteratively as

$$\tilde{\Phi}_1(A(\tau), k, \Delta) \in \mathtt{I} \oplus \underbrace{\bigoplus_{l_1=1}^{k} \mathcal{A}\Delta}_{\subseteq \mathtt{CH}(\mathcal{A})t},$$

$$\tilde{\Phi}_i(A(\tau), k, \Delta) \in \tilde{\Phi}_{i-1}(t, \Delta) \oplus \underbrace{\bigoplus_{l_i=1}^{k} \cdots \bigoplus_{l_1=1}^{l_2} \mathcal{A}^i\Delta^i}_{\subseteq \frac{1}{i!}\mathtt{CH}(\mathcal{A}^i)t^i =: \overline{\mathcal{M}}_i(t)},$$

where $\mathtt{CH}()$ is the convex hull operator, which ensures that the distributivity law can be applied.

Overapproximation of the State Transition Matrix

The expressions $\overline{\mathcal{M}}_i(t)$ are independent of Δ. For $\lim_{\Delta \to 0}$ we have that

Overapproximation of the state transition matrix

$$\Phi(A(\tau), t) \in \bigoplus_{i=0}^{\infty} \overline{\mathcal{M}}_i(t), \qquad \overline{\mathcal{M}}_i(t) = \frac{t^i}{i!} \mathtt{CH}(\mathcal{A}^i).$$

Overapproximation of the state transition matrix: time invariant case

$$\Phi(A, t) \in \left\{ \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \middle| A \in \mathcal{A} \right\}.$$

Overapproximation of the State Transition Matrix

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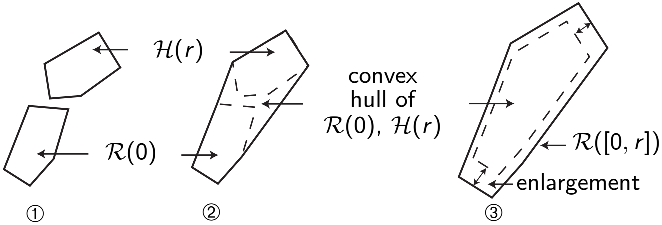
Overapproximation of the state transition matrix by a finite sum

$$\Phi_i(A(\tau), t) \in \bigoplus_{i=0}^{\eta} \overline{\mathcal{M}}_i(t) \oplus [-W(t), W(t)],$$

$W(t)$: remainder bound

Overview of Reachable Set Computation

1. Compute reachable set $\mathcal{H}(r)$ at time r when there is no input.
   done
2. Obtain convex hull of initial set $\mathcal{R}(0)$ and $\mathcal{H}(r)$.
   trivial
3. Enlarge reachable set to account for (1) uncertain inputs (*next slide*), (2) curvature of trajectories (*skipped*).
4. Continue with further time intervals $[kr, (k+1)r]$, $k \in \mathbb{N}$.



Input Solution

Removing the input

The differential equation $\dot{x}(t) = A(t)x(t) + u(t)$ can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & u(t) \\ 0 & 0 \end{pmatrix}}_{A_u(t)} \begin{pmatrix} x(t) \\ 1 \end{pmatrix}$$

...analogous proofs ...

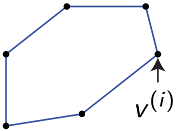
Reachable set due to the input

$$\mathcal{P}(t) = \bigoplus_{i=0}^{\eta} \left(\frac{t^{i+1}}{(i+1)!} \mathtt{CH}(\mathcal{A}^i \mathcal{U}) \right) \oplus \frac{t}{\eta+2} [-W(t), W(t)] \{|\mathcal{U}|\} .$$

Typical Types of Sets for Reachability Analysis

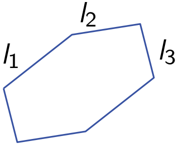
Polytopes: Convex hull of vertices

$$\left\{ \sum_{i=1}^{r_A} \alpha_i v^{(i)} \middle| v^{(i)} \in \mathbb{R}^n, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$



Zonotopes: Minkowski sum of line segments $l_i = [-1, 1] g^{(i)}$

$$\left\{ g^{(0)} + \sum_{i=1}^{\kappa_A} p_i g^{(i)} \middle| g^{(i)} \in \mathbb{R}^n, p_i \in [-1, 1] \right\}$$



Interval Vector

$$[\underline{a}, \overline{a}], \quad \forall i : \underline{a}_i \leq \overline{a}_i, \quad \underline{a}, \overline{a} \in \mathbb{R}^n.$$

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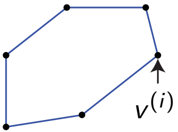

Zonotopes: Minkowski sum of line segments
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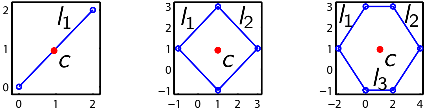
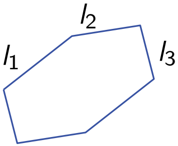




Interval Vector

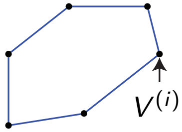
$$[\underline{a}, \overline{a}], \quad \forall i : \underline{a}_i \leq \overline{a}_i, \quad \underline{a}, \overline{a} \in \mathbb{R}^n.$$

Considered Matrix Sets for $\mathcal{A}$

Analogous definitions for matrix sets:

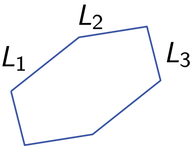
Matrix Polytopes: Convex hull of matrices

$$\left\{ \sum_{i=1}^{r_A} \alpha_i V^{(i)} \middle| V^{(i)} \in \mathbb{R}^{n\times n}, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$



Matrix Zonotopes: Minkowski sum of "matrix line segments" $L_i = [-1,1]G^{(i)}$

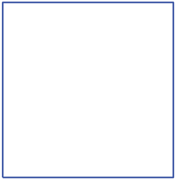
$$\left\{ G^{(0)} + \sum_{i=1}^{\kappa_A} p_i G^{(i)} \middle| G^{(i)} \in \mathbb{R}^{n\times n}, p_i \in [-1,1] \right\}$$



Interval Matrix

$$[\underline{A}, \overline{A}], \quad \forall i,j : \underline{A}_{ij} \leq \overline{A}_{ij}, \quad \underline{A}, \overline{A} \in \mathbb{R}^{n\times n}.$$

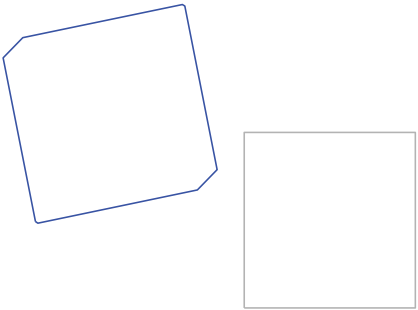
Reachability Algorithm



Compute $R([0, t_f])$

$$\mathcal{H}_0 = \mathtt{CH}(R(0) \cup \overline{\mathcal{M}}(r)R(0)) \oplus \mathcal{F}(r)R(0)$$
$$\mathcal{P}_0 = \mathcal{P}(r)$$
$$R_0 = \mathcal{H}_0 \oplus \mathcal{P}_0$$
for $k = 1 \dots t_f/r - 1$ **do**
$$R_k = \overline{\mathcal{M}}(r)R_{k-1} \oplus \mathcal{P}_0$$
end for
$$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$$

Reachability Algorithm



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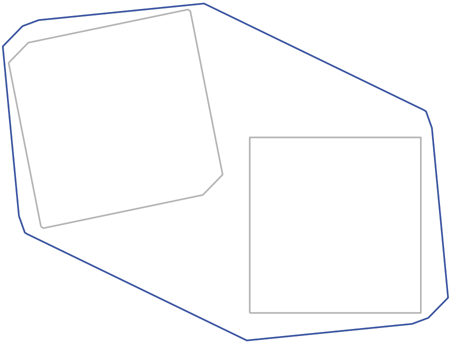
Reachability Algorithm



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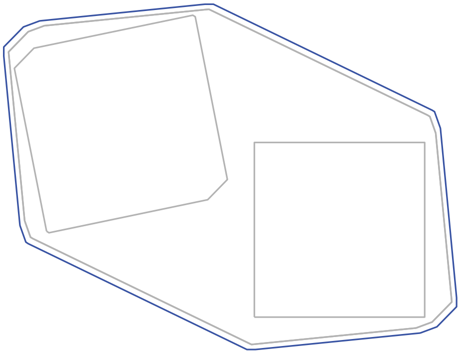

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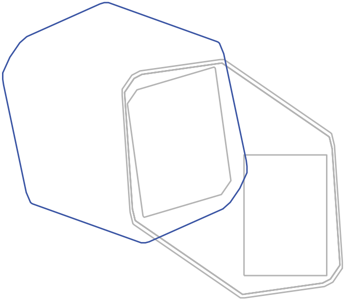
Reachability Algorithm



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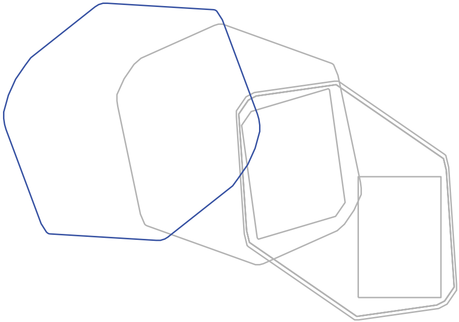
Reachability Algorithm



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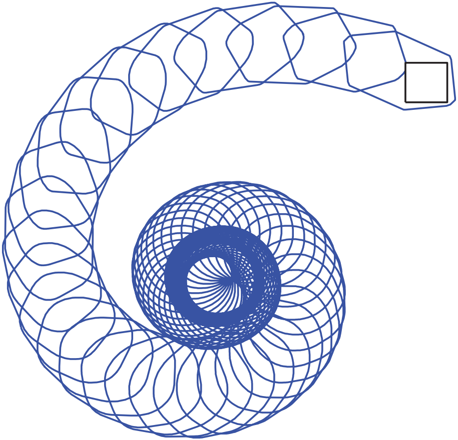
Reachability Algorithm



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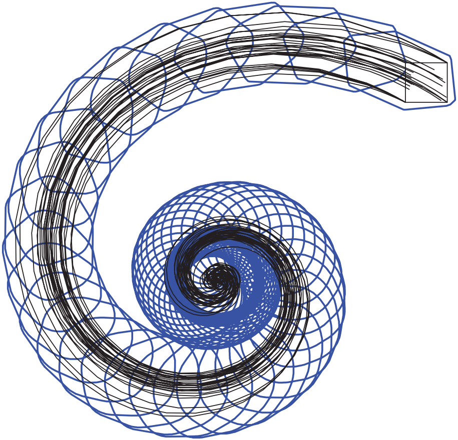
Reachability Algorithm



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end for

$$R([0, t_f]) = \bigcup_{k=1}^{t_f/r} R_{k-1}$$

Computation Times of Random Examples

- Random examples of linear systems for 100 time intervals are computed.
- The random system matrices might be unstable; but does not change computation time.

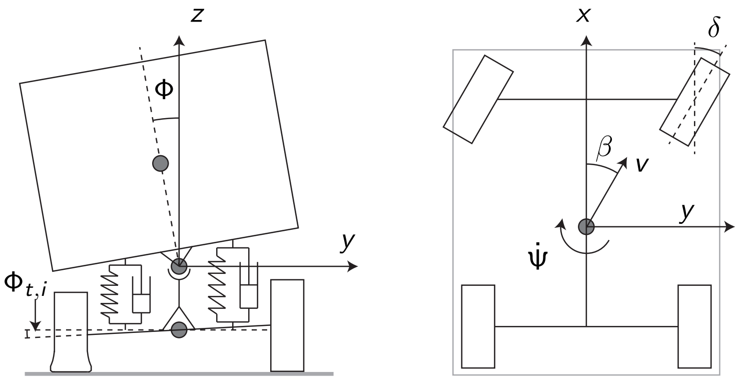
Table: Computation times in [s].

Dimension n	5	10	20	50	100
Interval matrix	0.10	0.12	0.33	0.82	3.64
Matrix zonotope $(\kappa = 1)$	0.13	0.17	0.60	2.65	8.72
Matrix zonotope $(\kappa = 2)$	0.18	0.30	1.13	4.73	18.77
Matrix zonotope $(\kappa = 4)$	0.34	0.68	2.60	18.07	98.70

κ: Number of generator matrices.

computed in MATLAB on an i7 Processor (1.6 GHz) and 6GB memory

Rollover Verification of a Truck



- Considered maneuver: Braking deceleration of $a_x = -0.7g$ (g: gravity constant); acceleration due to steering: $a_y \in [-0.4, 0.4]g$.
- Verification task: Can the vehicle roll over?
- state vector: $x = [\beta, \dot{\Psi}, \Phi, \dot{\Phi}, \Phi_{t,f}, \Phi_{t,r}, v, \int e(t)\, dt]^T$.

Dynamics of the Closed Loop System

truck dynamics (blue variables are states, red ones are inputs):

$$
\begin{aligned}
m x_7(\dot{x}_1 + x_2) - m_S h \dot{x}_4 &= Y_\beta x_1 + Y_{\dot{\psi}}(x_7) x_2 + Y_\delta \delta \\
-I_{xz}\dot{x}_4 + I_{zz}\dot{x}_2 &= N_\beta x_1 + N_{\dot{\psi}}(x_7) x_2 + N_\delta \delta \\
(I_{xx} + m_S h^2)\dot{x}_4 - I_{xz}\dot{x}_2 &= m_S g h x_3 + m_S h x_7(\dot{x}_1 + x_2) - k_f(x_3 - x_5) \\
&\quad - b_f(x_4 - \dot{x}_5) - k_r(x_3 - x_6) - b_r(x_4 - \dot{x}_6) \\
-r(Y_{\beta,f} x_1 + Y_{\dot{\psi},f} x_2 + Y_\delta \delta) &= m_{u,f}(r - h_{u,f}) x_7(\dot{x}_1 + x_2) + m_{u,f} g h_{u,f} x_5 \\
&\quad - k_{t,f} x_5 + k_f(x_3 - x_5) + b_f(x_4 - \dot{x}_5) \\
-r(Y_{\beta,r} x_1 + Y_{\dot{\psi},r} x_2) &= m_{u,r}(r - h_{u,r}) x_7(\dot{x}_1 + x_2) - m_{u,r} g h_{u,r} x_6 \\
&\quad - k_{t,r} x_6 + k_r(x_3 - x_6) + b_r(x_4 - \dot{x}_6) \\
\dot{x}_7 &= a_x.
\end{aligned}
$$

yaw controller:

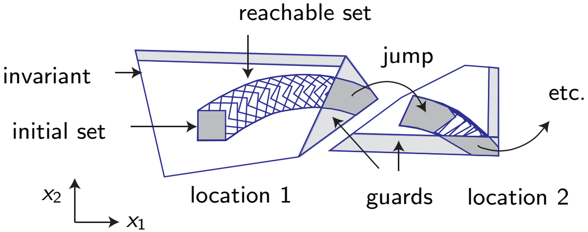
$$
\delta = k_1 e + k_2 \int e(t)\, dt, \quad e = \dot{\Psi}_d - \dot{\Psi} = \dot{\Psi}_d - x_2.
$$

velocity $x_7 \in$	$[10, 20]$ m/s	$[20, 30]$ m/s	$[30, \infty[$ m/s
controller gains	$k_1 = 0.4$ $k_2 = 1.5$	$k_1 = 0.5$ $k_2 = 2$	$k_1 = 0.6$ $k_2 = 2.5$

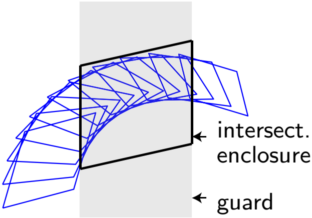
Standard Reachability Analysis of Hybrid Systems

Classical reachability analysis of hybrid systems

Reachable set computation is continued across discrete transitions using intersections with guard sets → Overapproximations due to intersections, overall complexity is not $\mathcal{O}(n^3)$ anymore.



(a) Reachable set of a hybrid system

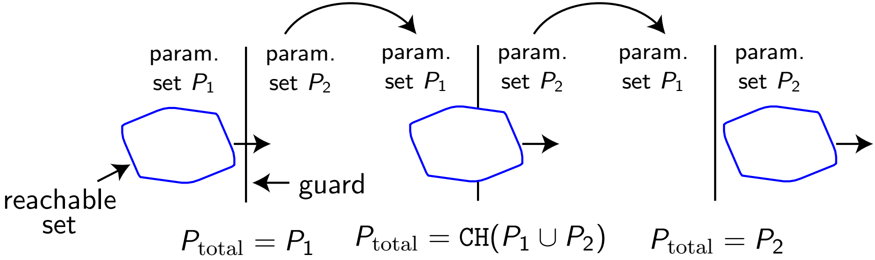


(b) Overapproximation due to guard intersection

Alternative Reachability Analysis of Hybrid Systems

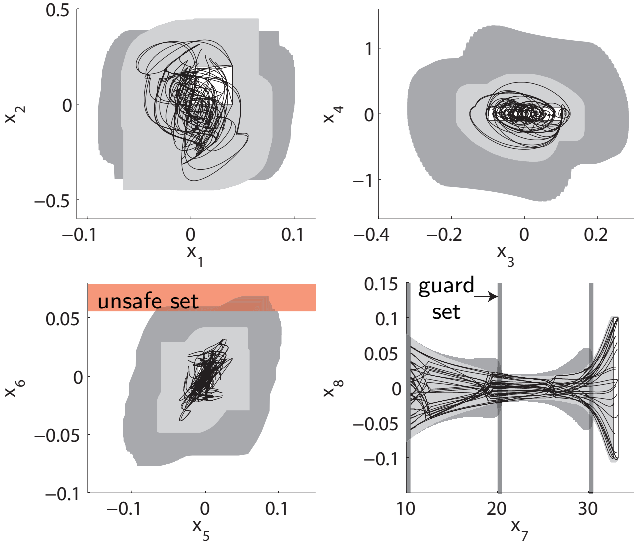
Reachability analysis using continuization

Reachable set is computed under a larger set of parameter uncertainties when intersecting several invariant sets.



- Only applicable if there are no jumps.
- Especially suited if the continuous dynamics does not change much.

Reachable Set of the Truck

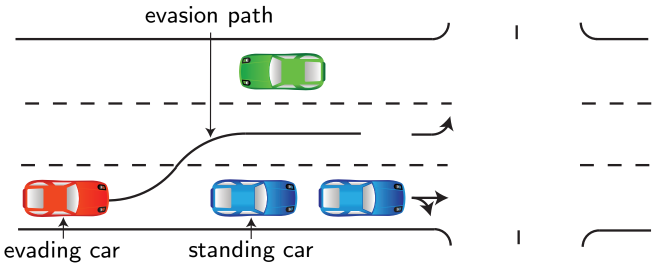


Verification of an Emergency Maneuver

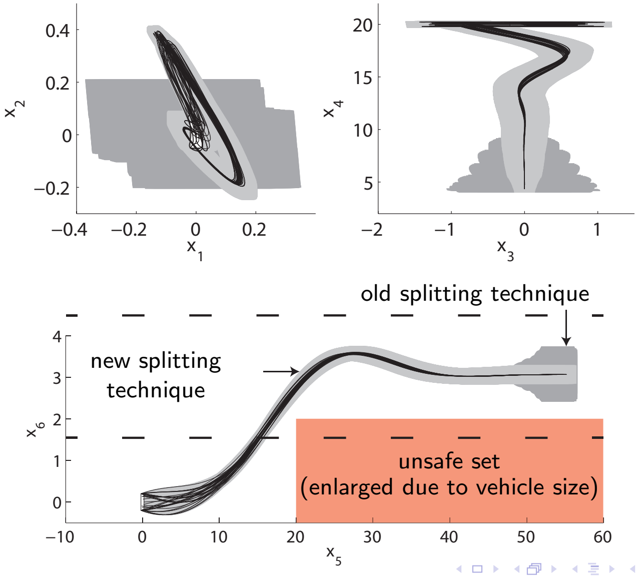
Motivation for automatic evasion maneuver

Crash is inevitable $\rightarrow$ vehicle automatically breaks, or steers, or does both. For velocities greater than $v = \sqrt{8a_{max}w}$, steering is more effective than braking.

a_{max}: maximum acceleration, w: width of the vehicle.



Reachable Set of the Evasive Maneuver



Next Step: Online Verification

Collaborator: Prof. John Dolan (Robotics Institute CMU)

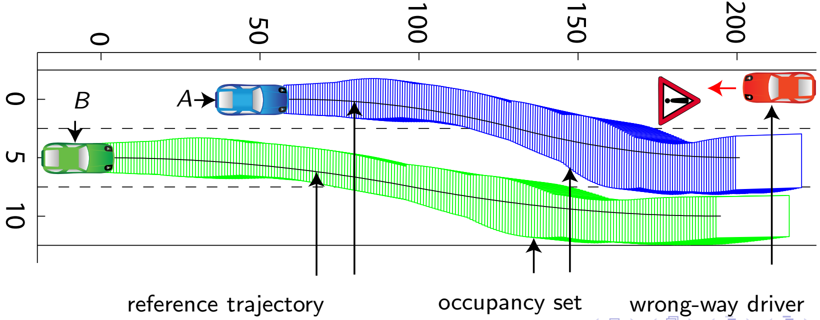
Case Study For Online Verification

Simplifications:

- constant velocity
- reference trajectory consists of arc segments

$\rightarrow$ System is linear.

Computation time including collision checks: 0.39 sec on desktop PC (AMD Athlon64 3700+) in MATLAB.



Conclusions

Reachability Analysis:

- Previous methods for the reachability analysis of LTI systems have been extended to uncertain linear time-varying systems.
- Approach scales well with the number of state variables ($\mathcal{O}(n^3)$).
- Continuization is promising for hybrid systems with similar continuous dynamics in adjacent locations.
- Result makes it possible to apply an alternative linearization approach $\rightarrow$ Further work required.

Automotive Applications:

- Cooperative intersection collision avoidance system (CICAS) with Toyota
- Verification of autonomous cars with the Robotics Institute at CMU.